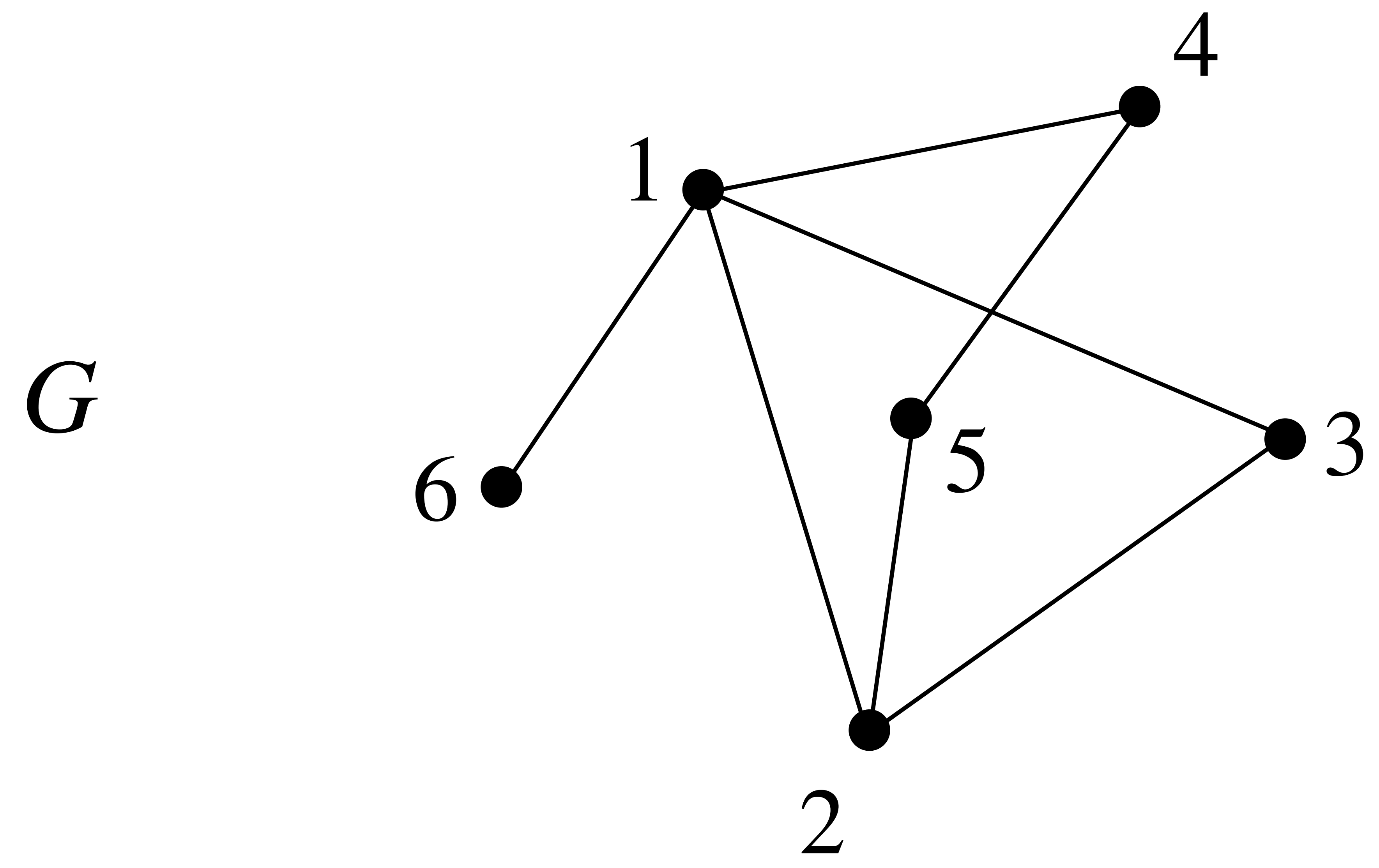
Lecture 29

Hamiltonian Cycles, Isomorphic Graphs

Hamiltonian Paths and Cycles

Definition: A path in a graph that contains all the vertices of the graph is called a **Hamiltonian Path**. A cycle in a graph that contains all the vertices of the graph is called a **Hamiltonian Cycle**.



Example: A Hamiltonian path in G is $\langle 6,1,4,5,2,3 \rangle$

A Hamiltonian cycle in $G \cup \{6,3\}$ is $\langle 6,1,4,5,2,3,6 \rangle$

Hamiltonian Cycles in Real-life

Scenario 1: A salesman has to visit several localities. He wants to start and end from the same locality, and further wants to visit all the localities exactly once.

- Consider localities as vertices and put an edge between two localities if they are connected.
- A hamiltonian cycle in the constructed graph gives a desired tour of localities.

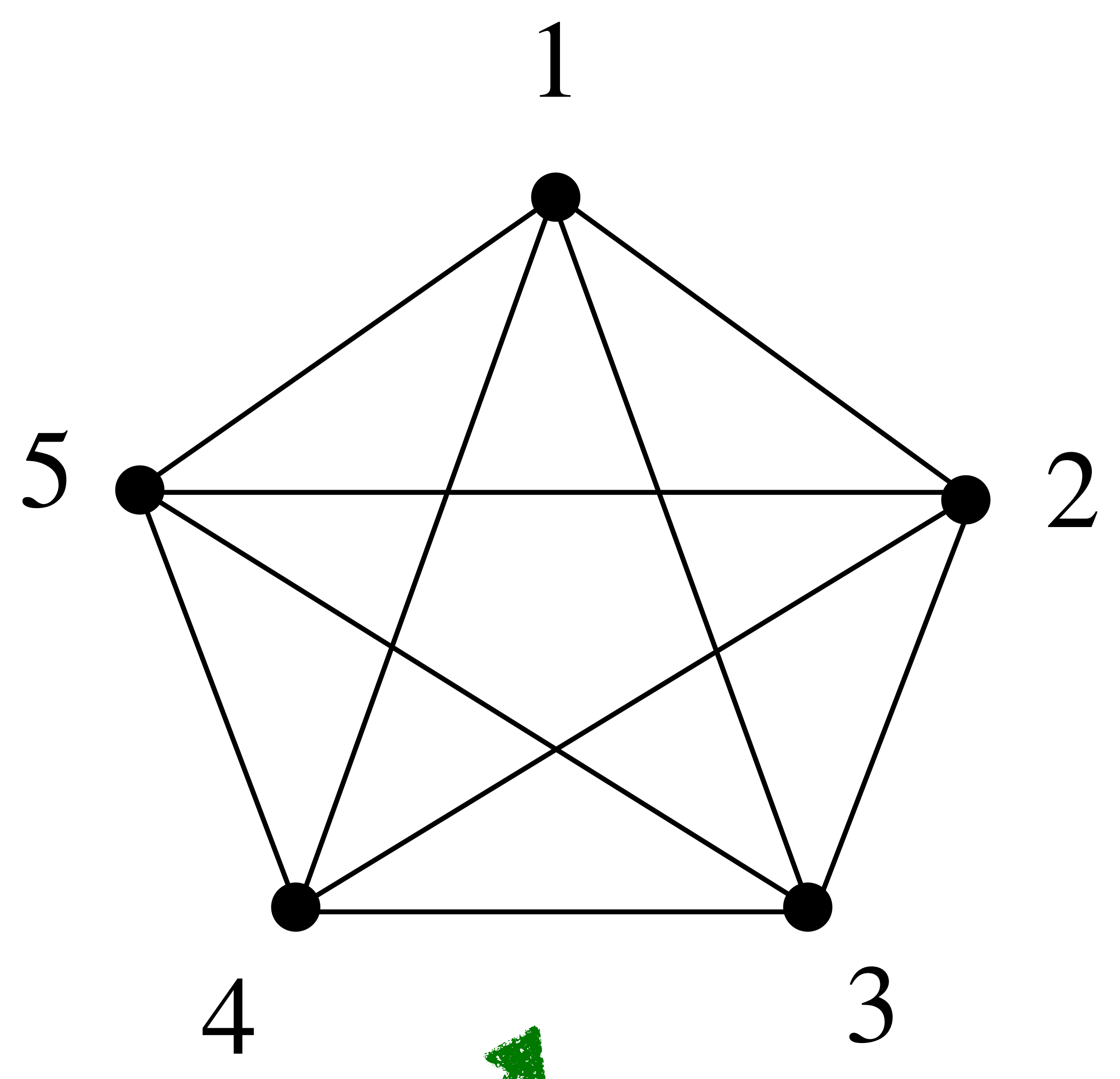
Scenario 2: You have several guests in your house and you want them to sit around a circular table such that each guest knows the people sitting next to them.

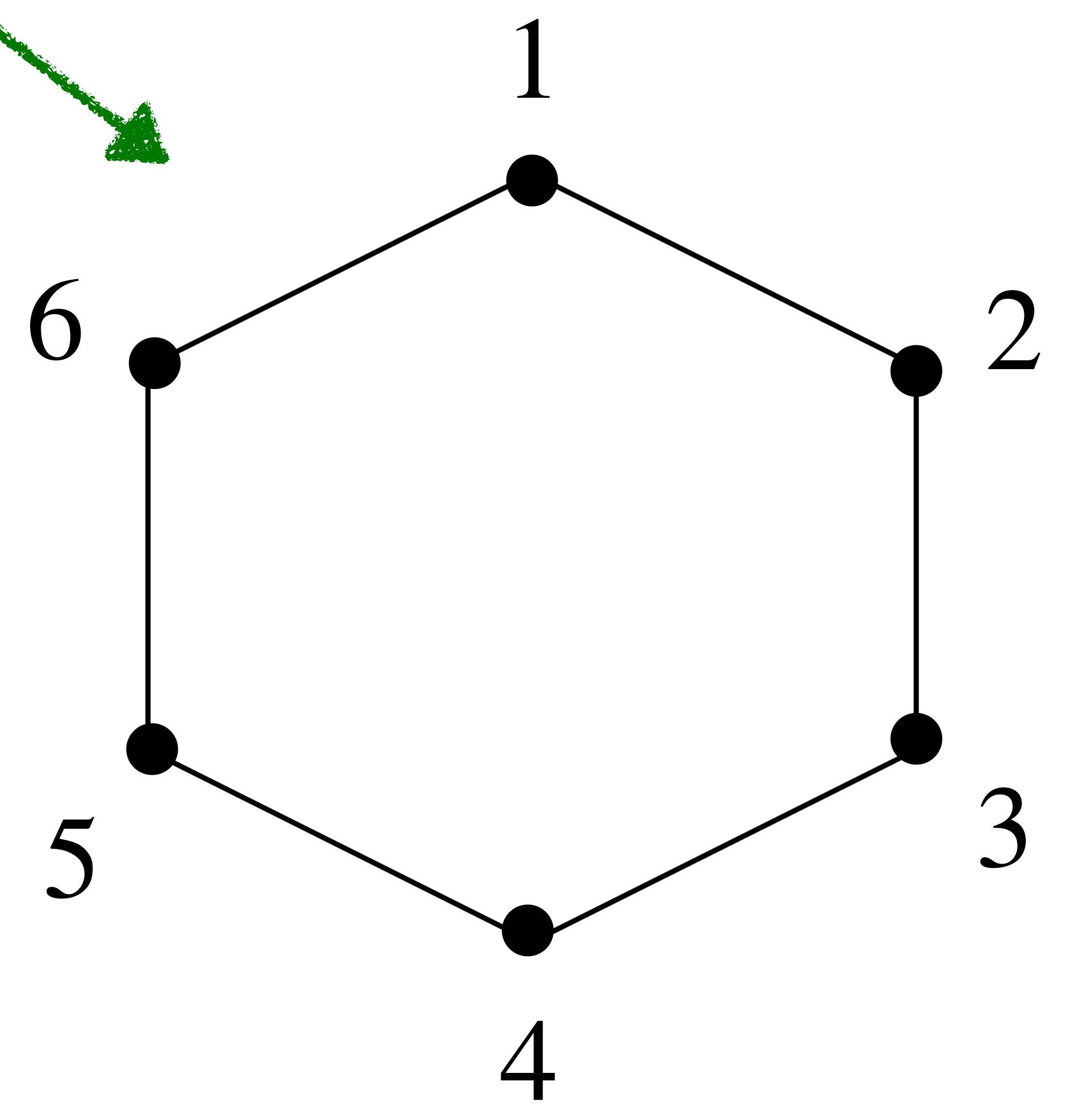
- Consider guests as vertices and put an edge between two guests if they know each other.
- A hamiltonian cycle in the graph provides an appropriate sitting.

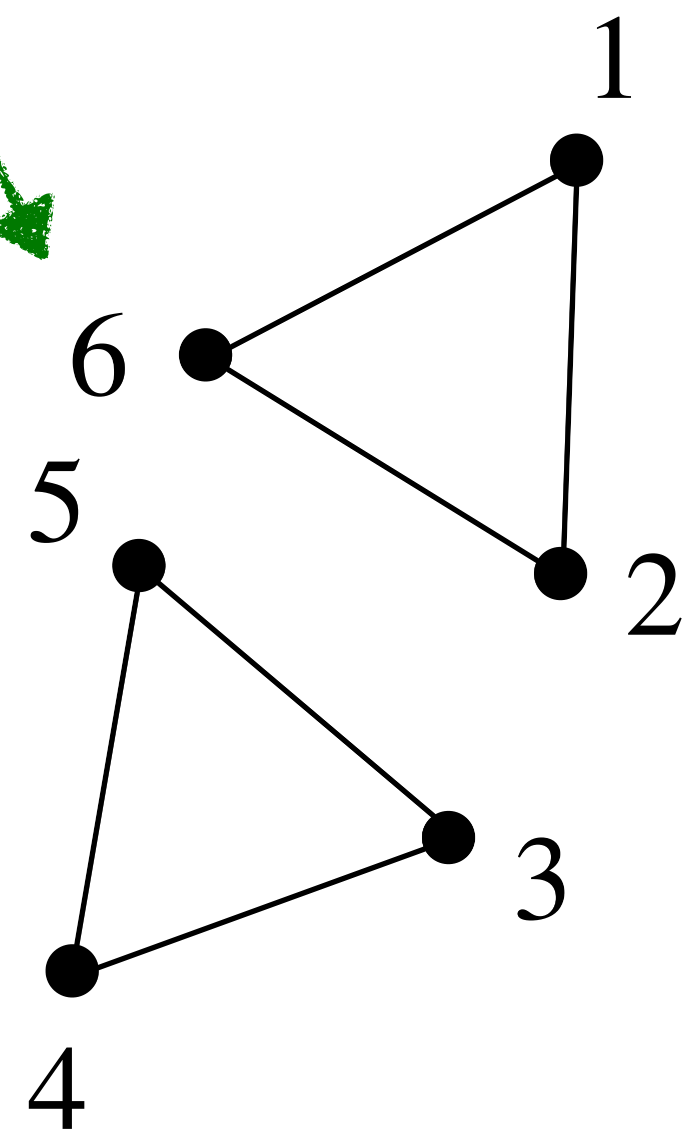
Testing for Hamiltonian Cycles

A hamiltonian cycle implies that every vertex has at least 2 degree.

At least 2 degree of every vertex does not imply existence of a hamiltonian cycle







If all the vertices are of degree $n - 1$, for $n \geq 3$, then graph has a hamiltonian cycle.

Can we decrease this number further?

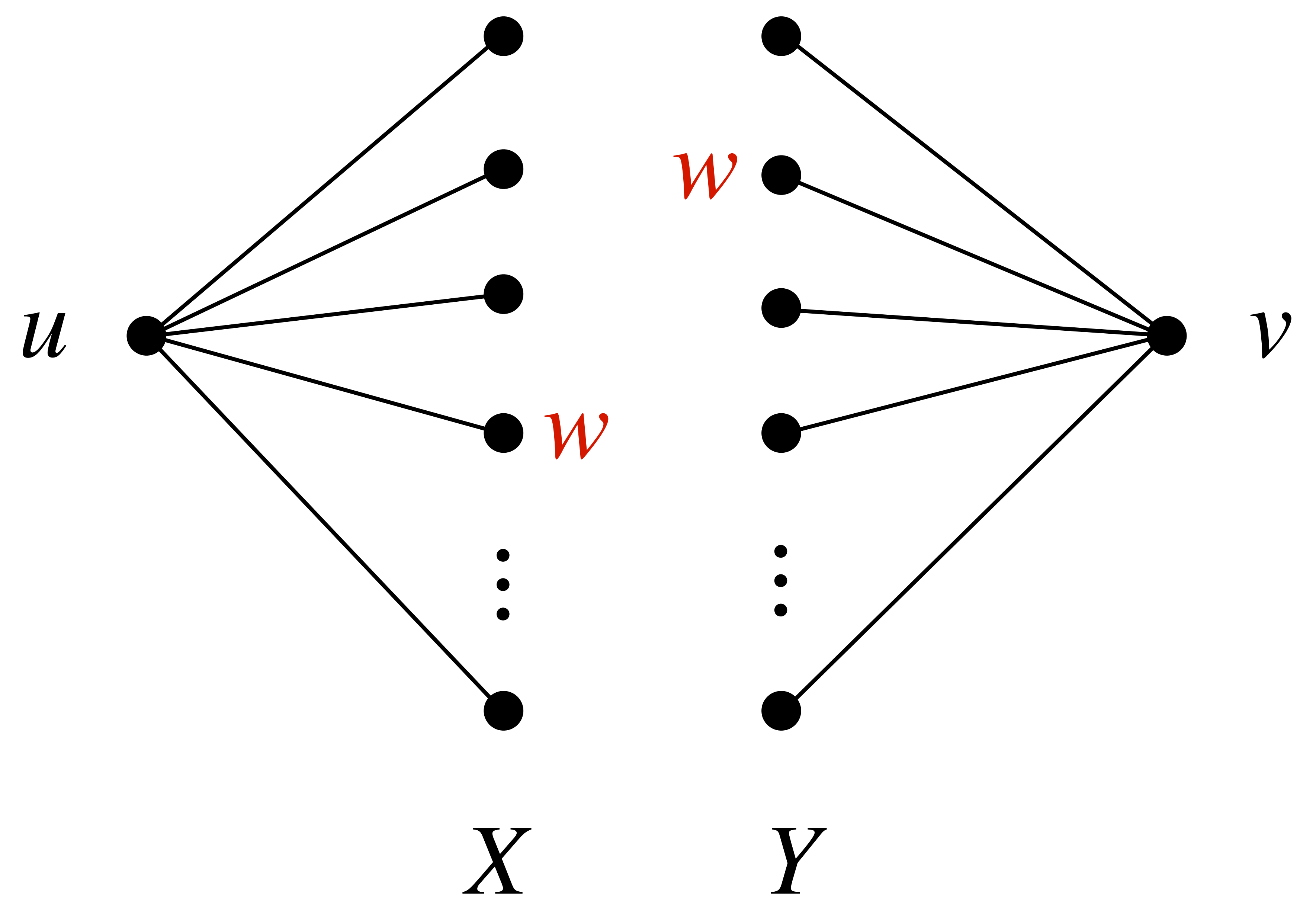
Dirac's Theorem

Theorem: Let G be a graph of $n \geq 3$ vertices. If all the vertices are of degree at least $n/2$, then G has a hamiltonian cycle.

Proof: Let's first ensure that G is connected.

Let u and v be any two distinct vertices. If they are neighbours, we are done.

If not:



Can X and Y be disjoint?

No, $\because$ otherwise $|\{u, v\}| + |X| + |Y| \geq n + 2$.

Therefore, X and Y have a common element w.

This implies that $\langle u, w, v \rangle$ is a path.

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Dirac's Theorem

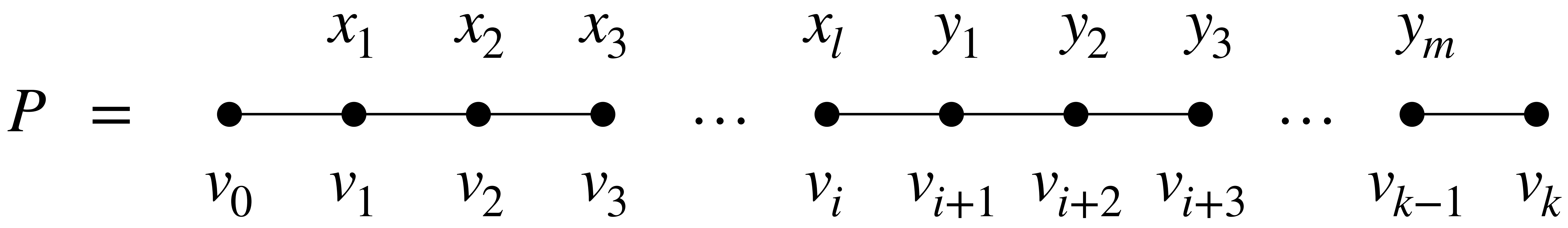
Let $P = \langle v_0, v_1, v_2, \ldots, v_k \rangle$ be a longest path in G.

Since P cannot be extended to a longer path all the neighbours of v_0 and v_k must lie on P.

Neighbours of v_0 = $\{x_1, x_2, \ldots, x_l\}$, where $l \geq n/2$.

Neighbours of v_k = $\{y_1, y_2, \ldots, y_m\}$, where $m \geq n/2$.

Can all neighbours of v_0 precede all neighbours of v_k in P?



Not possible as it implies that there are at least $1 + n/2 + n/2 + 1 = n + 2$ vertices in P.

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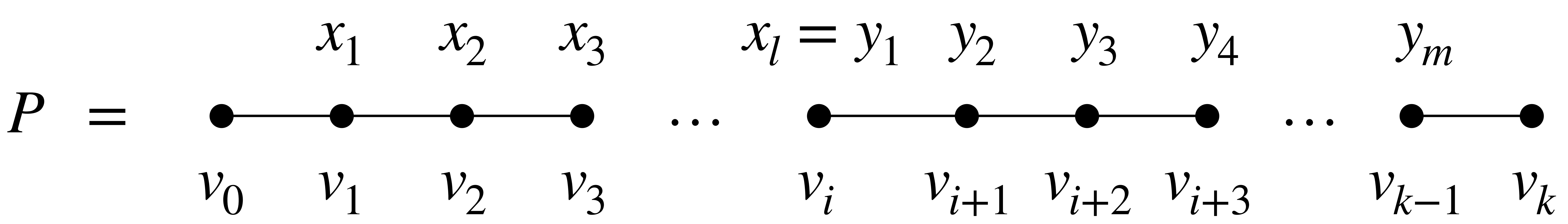
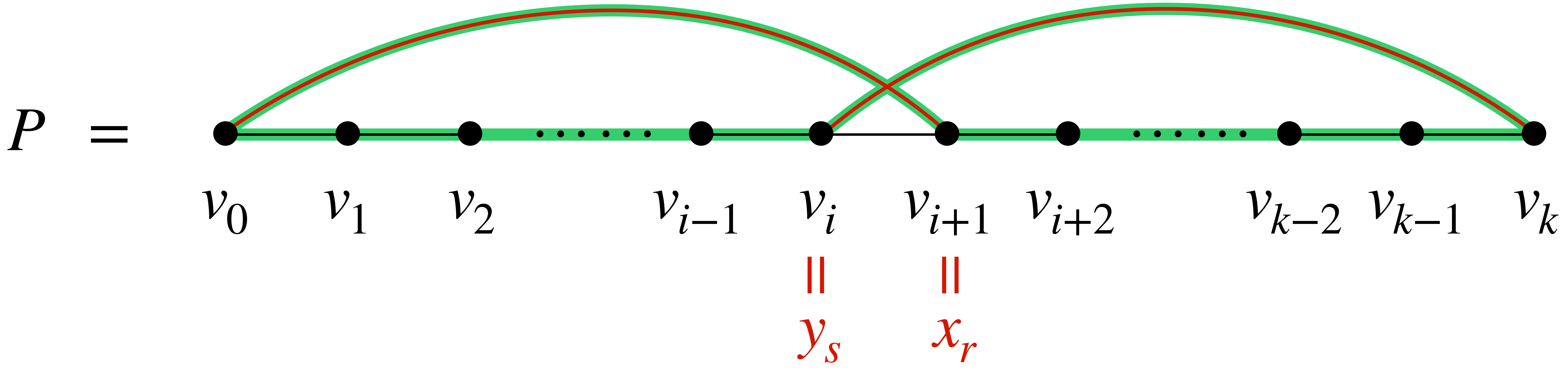
Dirac’s Theorem

Due to similar reason, the following is also not possible.



Hence, some neighbour of v_k will definitely precede some neighbour of v_0 in P.



This gives us a cycle C of length $k + 1$.

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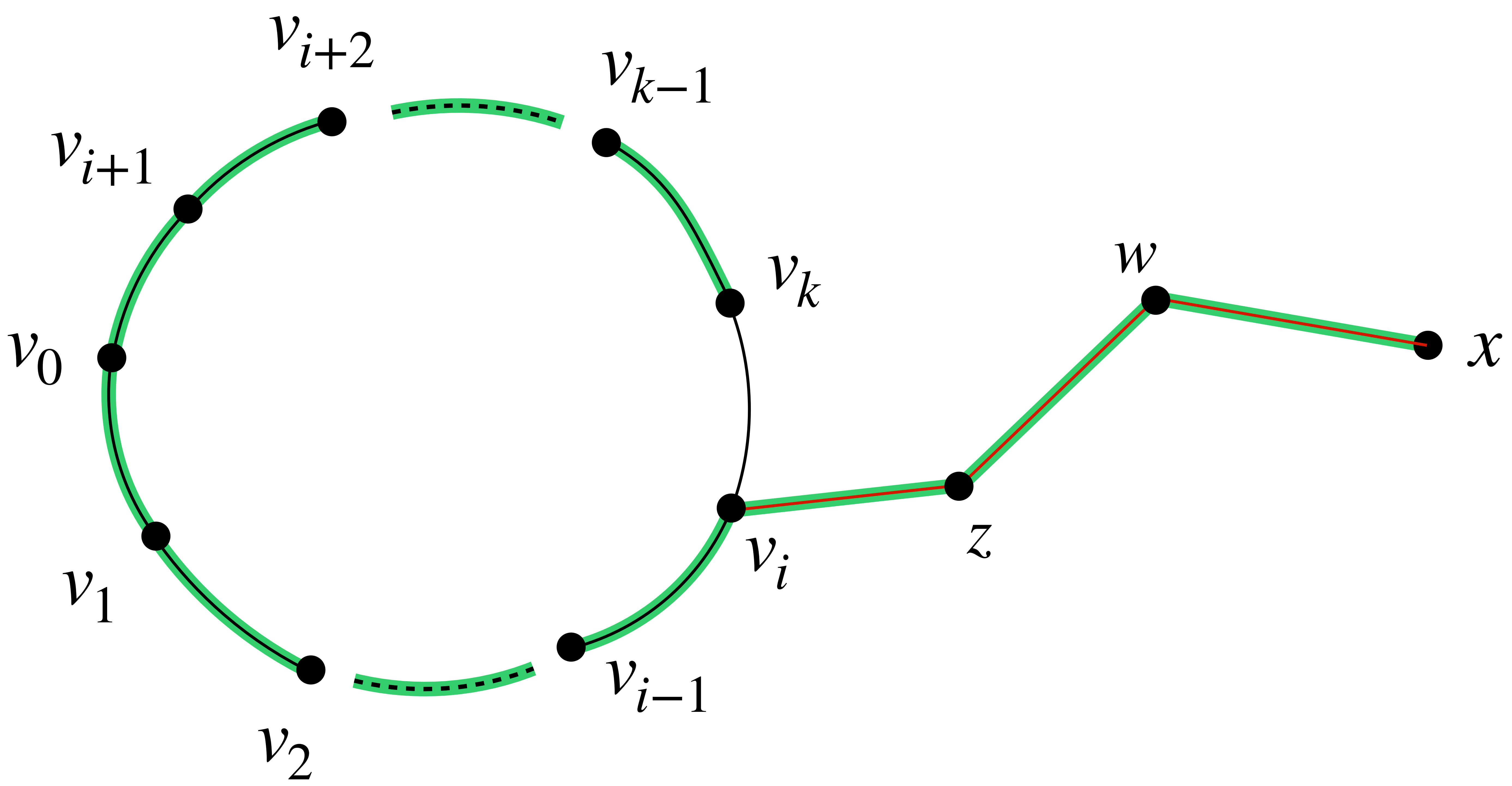
Dirac's Theorem

We claim that the cycle C is a hamiltonian cycle. If not, then $\exists\, x$ such that $x \notin C$.

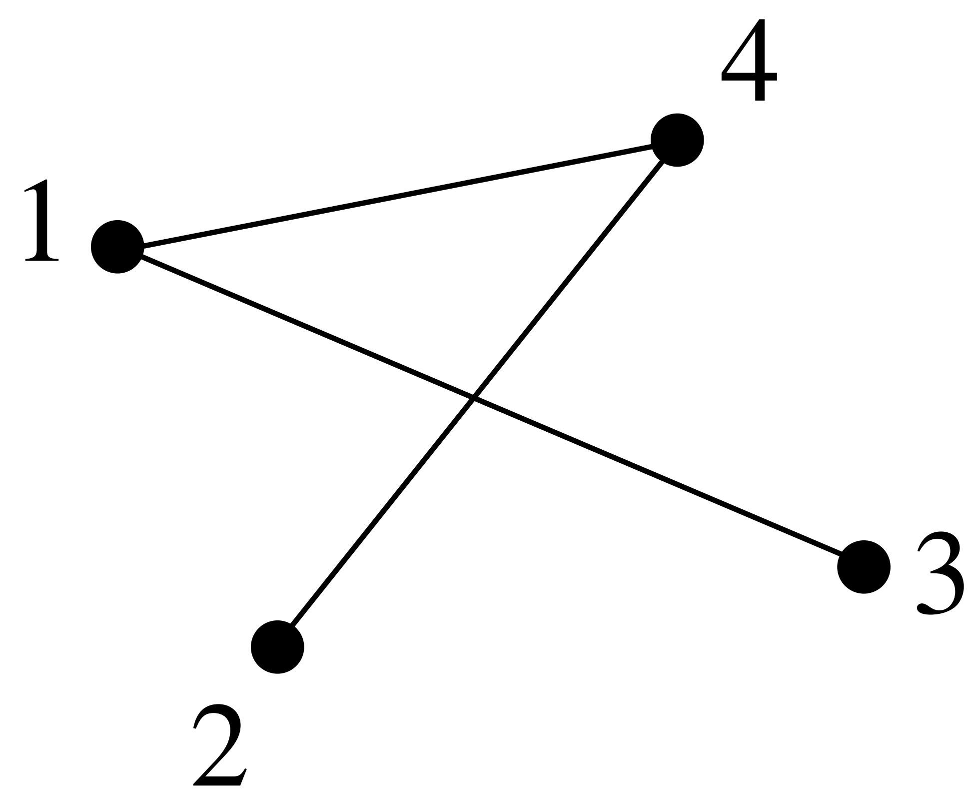


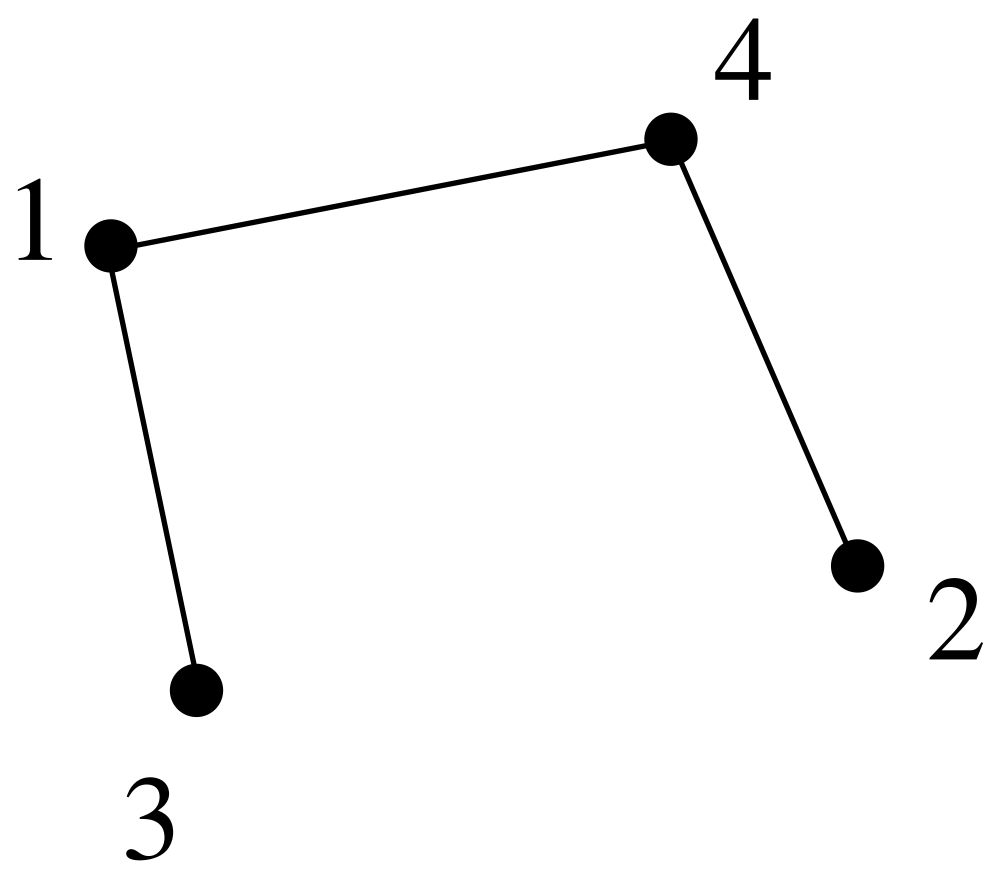
Because G is connected there must be a path from x to some vertex in C, say v_i.

This implies a path $\langle x, w, z, v_i, v_{i-1}, \ldots, v_0, v_{i+1}, \ldots, v_{k-1}, v_k \rangle$ longer than P, a contradiction. ■

When are two graphs same?

Are the following graphs same?

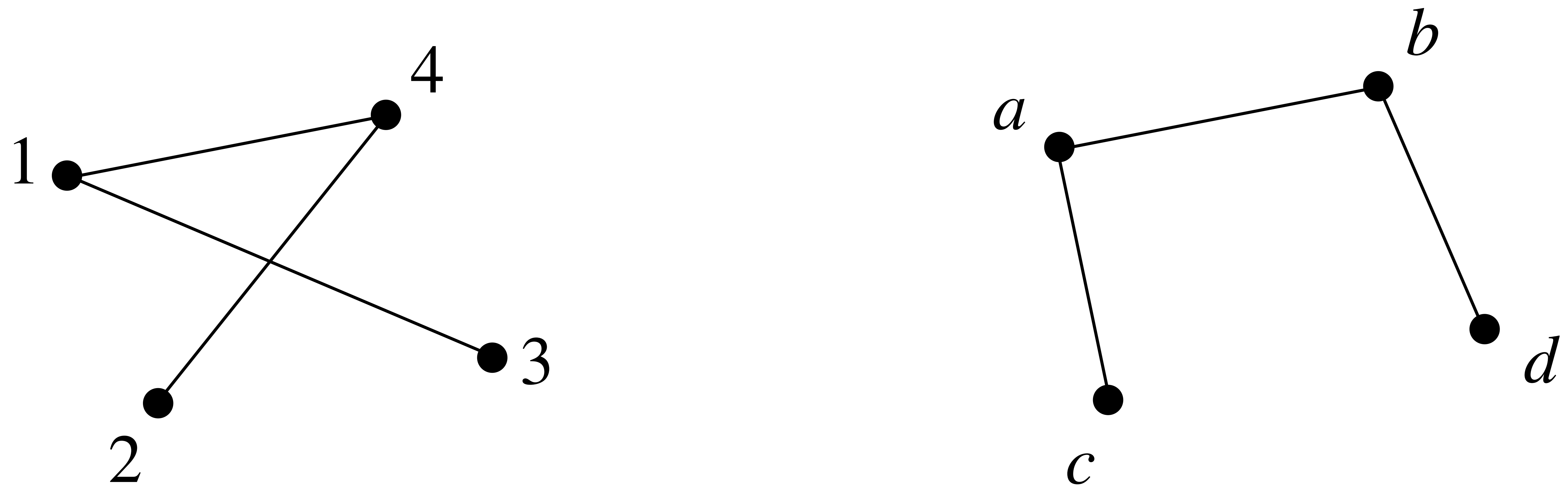




Yes. Different drawings but same graphs.

When are two graphs same?

Are the following graphs same?



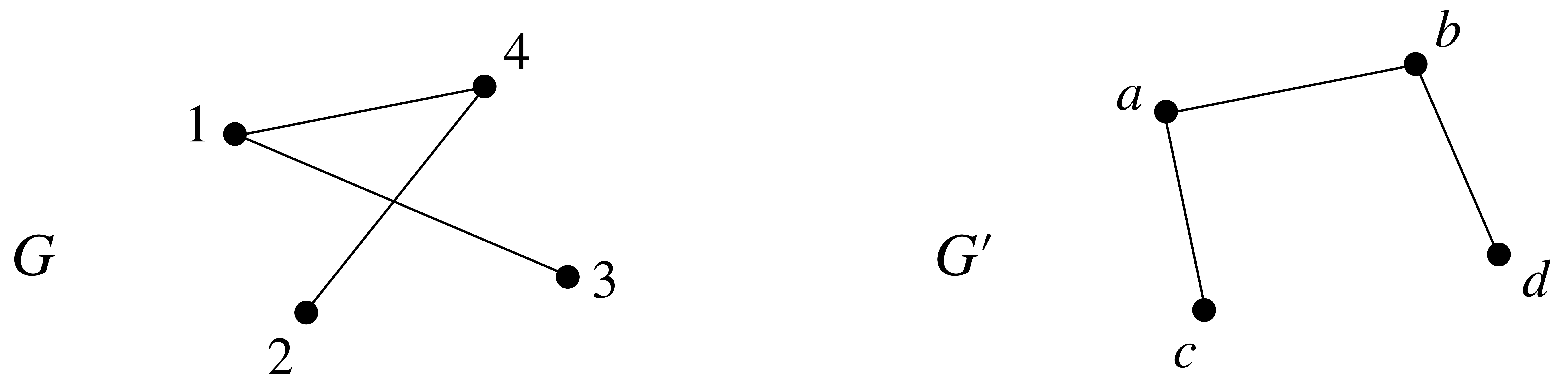
Different, but not completely unrelated. If we drop the labels, they are same.

We call such graphs **Isomorphic Graphs**.

Isomorphic Graphs

Definition: We say that graphs $G = (V, E)$ and $G' = (V', E')$ are **isomorphic** if there is a bijection $f : V \to V'$ such that two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in G'. The bijection f is called **isomorphism**.

Example:



Isomorphism f between G and G':

$$f(1) = a,\ f(2) = d,\ f(3) = c,\ f(4) = b.$$